“Number Rules the Universe”
Number Theory through the Ages

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Pythagoras and his theorem

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The Pythagorean Theorem

If \( \triangle ABC \) is a right triangle whose legs have lengths \( a \) and \( b \) and whose hypotenuse has length \( c \), then

\[
a^2 + b^2 = c^2.
\]

This is often called the Pythagorean equation.
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Pythagorean triples

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Plimpton 322
Babylonian tablet
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5. Plimpton 322 (a Babylonian tablet, c. 1800 BC)
   - Row 1: $a = 120$, $b = 119$ and $c = 169$
   - Row 3: $a = 4800$, $b = 4601$ and $c = 6649$
   - Row 4: $a = 13500$, $b = 12709$ and $c = 18541$
Pythagorean triples

Pythagoras’ parametrization of Pythagorean triples:

\[ n^2 + \left( \frac{n^2 - 1}{2} \right)^2 = \left( \frac{n^2 + 1}{2} \right)^2 \]

where \( n \) is any odd number.
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Pythagorean triples

Plato’s parametrization of Pythagorean triples:

\[(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2\]

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where $n$ is any number.

Euclid’s parametrization of Pythagorean triples (Euclid X.29):

$$(pq)^2 + \left(\frac{p^2 - q^2}{2}\right)^2 = \left(\frac{p^2 + q^2}{2}\right)^2$$

where $p, q$ are two numbers of the same parity and $p > q$. 
Pierre de Fermat (1601–1665)

“It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvelous demonstration of this proposition that this margin is too narrow to contain.”

— Fermat, c. 1630
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Fermat's Last Theorem

*The equation*

\[ x^n + y^n = z^n \]

*has no solutions in positive whole numbers whenever* \( n \geq 3 \).
The end of a 350-year-old problem

To ensure (1) holds we use Hilbert irreducibility:

\[ \exists f, \text{ a finite collection of irreducible polynomials}, f_i(x, t) \in \mathbb{Q}(t)[x] \]

Each \( f_i \) for each one.

Pick a \( p \not\equiv 5 \pmod{4} \) has no root mod \( p \).

Then pick a non-constant \( \eta \in \mathbb{Q} \) which is \( p \)-adically close to the original \( E_\eta \)

So \( t \rightarrow E' \)

\[ E_{\eta} \rightarrow E' \]

By then
Leonhard Euler (1707–1783)

“It has seemed to many Geometers that this theorem [Fermat’s Last ‘Theorem’] may be generalized. Just as there do not exist two cubes whose sum or difference is a cube, it is certain that it is impossible to exhibit three biquadrates whose sum is a biquadrate, but that at least four biquadrates are needed if their sum is to be a biquadrate, although no one has been able up to the present to assign four such biquadrates. In the same manner it would seem impossible to exhibit four fifth powers whose sum is a fifth power, and similarly for higher powers.” — Euler, 1769
My work with Hillsdale senior Jonathan Gregg

Last spring I began work with Jon Gregg on a problem closely related to Euler’s Conjecture. Supported by a LAUREATES fellowship, we explored when equations of the form

\[ x_1^n + x_2^n + \cdots + x_k^n = y^n \]

have solutions in positive whole numbers.
Solutions to $x^3 + y^3 + z^3 = t^3$

By an iterative process, we found several primitive solutions to the equation

$$x^3 + y^3 + z^3 = t^3.$$ 

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<th>$z$</th>
<th>$t$</th>
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Parametric Solutions to Other Equations

We found a two-parameter solution

$$(6a^3 + b^3)^3 = (6a^3 - b^3)^3 + 2(b^3)^3 + (6a^2 b)^3$$

and a three-parameter solution

$$(9a^3 + b^3 + c^3)^3 = (9a^3 + b^3 - c^3)^3 + (9a^3 - b^3 + c^3)^3 + (-9a^3 + b^3 + c^3)^3 + (6abc)^3$$

to the equation $x^3 + y^3 + z^3 + w^3 = t^3$. 
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to the equation \(x^3 + y^3 + z^3 + w^3 = t^3\).

We also found a four-parameter solution to the equation

\[x_1^8 + x_2^8 + \cdots + x_8^8 = y_1^8 + y_2^8 + \cdots + y_9^8,\]

but this slide is not large enough to contain it.
The Pythagorean Theorem
Proofs of the Pythagorean Theorem

Behold!

\[
\begin{align*}
\text{Left:} & & \text{Right:} \\
& a & & a \\
& b & & b \\
& a & & a \\
& b & & b
\end{align*}
\]

\[
\text{Left:} & & \text{Right:} \\
& a + b & & a + b
\]
Elliptic Curves