Chapter 9. Problem 58. Suppose that a group $G$ has a subgroup of order $n$. Prove that the intersection of all subgroups of $G$ of order $n$ is a normal subgroup of $G$.

**Solution:** Let $G$ be a group with a subgroup of order $n$, and let $\{ H_n \}$ be the collection of all such subgroups. Let $N = \cap_n H_n$. Then from previous work, we know $N \leq G$. Now, let $x \in G$ and consider the set $x^{-1}N x$. For each $x$, this is the inner automorphism of $G$ induced by $x^{-1}$, acting on the subgroup $N$. Being an automorphism, $x^{-1}H_n x$ will be another subgroup of $G$ of order $n$. So we have $x^{-1}N x = x^{-1}(\cap_n H_n)x = \cap_n (x^{-1}H_n x)$, and since for each $x^{-1}H_n x$, $x^{-1}H_n x \in \{ H_n \}$, we know $\cap_n (x^{-1}H_n x) \supseteq N$. That is $x^{-1}N x \supseteq N$. Multiplying through on the left by $x$ and on the right by $x^{-1}$, we have $N \supseteq xNx^{-1}$, so by the normal subgroup test, $N \triangleleft G$.

Chapter 10. Problem 6. Let $G$ be the group of all polynomials with real coefficients under addition. For each $f$ in $G$, let $\int f$ denote the antiderivative of $f$ that passes through the point $(0,0)$. Show that the mapping $f \rightarrow \int f$ from $G$ to $G$ is a homomorphism. What is the kernel of this mapping? Is this mapping a homomorphism if $\int f$ denotes the antiderivative of $f$ that passes through $(0,1)$?

**Solution:** Let $f$ and $h$ be polynomials with real coefficients. Since $\int f$ and $\int h$ each pass through the point $(0,0)$, their sum will also pass through $(0,0)$: let $F = \int f$ and $H = \int h$. Then $F(0) = 0$ and $H(0) = 0$, so $(F + H)(0) = 0$. We know from calculus that the antiderivative of $f + h$ is the same as the sum of the antiderivatives of $f$ and $h$. That is, $\int f + h = F + H$, since we have already seen that $F + H$ passes through $(0,0)$. Let $f_0$ denote the zero polynomial ($f_0(x) = 0 \forall x$). Then the kernel of this map is $\ker \int = \{ f \in G : \int f = f_0 \} = \int^{-1}(f_0) = \{ \frac{d}{dx}f_0 \} = \{ f_0 \}$, since $\int^{-1} = \frac{d}{dx}$.

Now, consider the image of the zero polynomial, $f_0$, under the mapping where $\int f$ denotes the antiderivative of $f$ that passes through $(0,1)$. Then $\int f_0 = 1$, the constant function. But $f_0 + f_0 = f_0$, so if this map were a homomorphism, we would have $\int f_0 = \int(f_0 + f_0) = \int f_0 + \int f_0$. That is, $1 = 1 + 1 = 2$, which is obviously not true. Since this map does not preserve the operation, it is not a homomorphism.

Chapter 10. Problem 40. (Third Isomorphism Theorem) If $M$ and $N$ are normal subgroups of $G$ and $N \leq M$, prove that $(G/N)/(M/N) \approx G/M$.

**Solution:** First, define the map $\phi : G/N \rightarrow G/M$ by $\phi(gN) = gM$. We will show that $\phi$ is a well defined homomorphism, that $\phi(G/N) = G/M$ (i.e., that $\phi$ is onto), and that $\ker \phi = M/N$.

Now, let $g, g' \in G$ with $gN = g'N$. Then $N = g^{-1}g'N$, hence $g^{-1}g' \in N$. Since $N \leq M$, this means $g^{-1}g' \in M$, as well. So, $\phi(g^{-1}g'N) = g^{-1}g'M = M$. Then, multiplying through by $g$, we have $gM = gM$, or $\phi(g'N) = \phi(gN)$. Hence, $\phi$ is well defined.

Let $g_1 N, g_2 N \in G/N$. Then $\phi(g_1 N g_2 N) = \phi(g_1 g_2 N) = g_1 g_2 M = g_1 M g_2 M = \phi(g_1 N) \phi(g_2 N)$, hence $\phi$ is a homomorphism.

Let $g \in G$, then $gM \in G/M$. We know that there is a coset of $N$ in $G$ containing $g$, and that we can call this coset $gN$. By the definition of our map, we have $\phi(gN) = gM$, hence $\phi$ is onto. That is, $\phi(G/N) = G/M$. 

Finally, to see that $\ker \phi = M/N$, we consider their definitions: \( \ker \phi = \{gN|\phi(gN) = eM\} = \{gN|gM = M\} = \{gN|g \in M\} \). Meanwhile, we know that $M/N = \{mN|m \in M\}$, which is identical to the set $\ker \phi$, so we have $\ker \phi = M/N$.

Finally, by the First Isomorphism Theorem, we conclude that $(G/N)/(M/N) \cong G/M$.

Chapter 10. Problem 44. Let $N$ be a normal subgroup of a finite group $G$. Use the theorems of this chapter to prove that the order of the group element $gN$ divides the order of $g$.

**Solution**: Define the map $\phi : G \to G/N : g \mapsto gN$. Then, for $g_1, g_2 \in G$, $\phi(g_1g_2) = g_1g_2N = g_1Ng_2N = \phi(g_1)\phi(g_2)$. Hence, $\phi$ is a homomorphism from $G$ to $G/N$. So, by Theorem 10.1, part 3, $|\phi(g)|$ divides $|g|$. That is, $|gN|$ divides $|g|$.

Chapter 10. Problem 52. Let $\alpha$ and $\beta$ be group homomorphisms from $G$ to $\overline{G}$ and let $H = \{g \in G|\alpha(g) = \beta(g)\}$. Prove or disprove that $H$ is a subgroup of $G$.

**Solution**: Claim: $H$ is a subgroup of $G$. We will use the two step subgroup test to prove this claim. First, suppose that $a, b \in H$. Then $\alpha(a) = \beta(a)$ and $\alpha(b) = \beta(b)$. And since $\alpha$ and $\beta$ are homomorphisms, $\alpha(ab) = \alpha(a)\alpha(b) = \beta(a)\beta(b) = \beta(ab)$. So we have $ab \in H$ whenever $a, b \in H$. Now, for $a \in H$, we know that $\alpha(a) = \beta(a)$ and by Theorem 10.1, part 2, $\alpha(a^{-1}) = [\alpha(a)]^{-1} = [\beta(a)]^{-1} = \beta(a^{-1})$. Hence $a^{-1} \in H$ whenever $a \in H$, and so $H \leq G$. 