Chapter 5. Problem 16. Associate an even permutation with the number +1 and an odd permutation with the number −1. Draw an analogy between the result of multiplying two permutations and the result of multiplying their corresponding numbers +1 or −1.

Solution: Since we define a permutation as even or odd based on its expression as the product of 2-cycles, we see that the parity of a product of permutations depends on the parities of the permutations we are multiplying. In fact, if we have two permutations, \( \alpha \) and \( \beta \), which we have written as the products of \( a \) and \( b \) 2-cycles, respectively, then we can write their product, \( \alpha \beta \) as the product of \( a + b \) 2-cycles. Hence, the product of two even permutations is an even permutation, of two odds will be even, and of an even and an odd will be odd. Analogously, multiplying +1 with itself or \(-1\) with itself yields +1, while multiplying +1 and \(-1\) (in either order) yields \(-1\).

\[
\begin{array}{c|cc}
\cdot & +1 & -1 \\
+1 & +1 & -1 \\
-1 & -1 & +1 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\cdot & \text{even} & \text{odd} \\
\text{even} & \text{even} & \text{odd} \\
\text{odd} & \text{odd} & \text{even} \\
\end{array}
\]

Chapter 5. Problem 18. Let \( \alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix} \) and \( \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix} \).

Write \( \alpha \), \( \beta \) and \( \alpha \beta \) as

a. products of disjoint cycles,

b. products of 2-cycles.

Solution: Notice that \( \alpha \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 1 & 3 & 5 \end{bmatrix} \). Thus,

a. \( \alpha = (12345)(678) \), \( \beta = (23847)(56) \), and \( \alpha \beta = (12485736) \).

b. \( \alpha = (43)(42)(41)(45)(76)(78) \), \( \beta = (48)(43)(42)(47)(56) \), and \( \alpha \beta = (37)(35)(38)(34)(32)(31)(36) \).

Chapter 5. Problem 22. Let \( \alpha \) and \( \beta \) belong to \( S_n \). Prove that \( \alpha^{-1} \beta^{-1} \alpha \beta \) is an even permutation.

Solution: Since we know that \( \alpha \) can be written as a product of 2-cycles, we can easily define its inverse as the product of the same 2-cycles, composed in the opposite order. In symbols: if \( \alpha = \gamma_1 \gamma_2 \cdots \gamma_s \) then \( \alpha^{-1} \) can be written \( \alpha^{-1} = \gamma_s \gamma_{s-1} \cdots \gamma_1 \). Since both \( \alpha \) and \( \alpha^{-1} \) can be written as the product of \( s \) 2-cycles, they are either both even or both odd (By Theorem 5.5 and the definition). Hence, if \( s \) is even, then both \( \alpha \) and \( \alpha^{-1} \) must be even. Likewise, if \( s \) is odd, then so are \( \alpha \) and \( \alpha^{-1} \). Similarly for \( \beta \), \( \beta \) and \( \beta^{-1} \) must either both be even or both be odd. We see that the product \( \alpha^{-1} \beta^{-1} \alpha \beta \) is either the product of 4 even permutations, 4 odd permutations, or 2 even and two odd permutations. Using the idea from Problem 16, we see that in any of these cases, we end up with an even permutation.
Chapter 5. Problem 36. In \(S_4\), find a cyclic subgroup of order 4 and a non-cyclic subgroup of order 4.

**Solution:** Let \(\sigma = (1234)\). Then \(\sigma^2 = (13)(24), \sigma^3 = (1432), \) and \(\sigma^4 = \epsilon\). Hence, \(\langle \sigma \rangle = \{\epsilon, \sigma, \sigma^2, \sigma^3\}\) is a cyclic subgroup of order 4.

Now, let \(\alpha = (12)\) and \(\beta = (34)\). Then \(\alpha = \alpha^{-1}, \beta = \beta^{-1}, \alpha \beta = (12)(34)\) and \((\alpha \beta)^{-1} = \beta^{-1} \alpha^{-1} = \beta \alpha = \alpha \beta\), since \(\alpha\) and \(\beta\) are disjoint 2-cycles. We can verify that the set \(\{\epsilon, \alpha, \beta, \alpha \beta\}\) is a non-cyclic subgroup by examining its Cayley Table, shown below.

<table>
<thead>
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<th></th>
<th>(\epsilon)</th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(\alpha \beta)</th>
</tr>
</thead>
<tbody>
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<td>(\epsilon)</td>
<td>(\epsilon)</td>
<td>(\alpha)</td>
<td>(\beta)</td>
<td>(\alpha \beta)</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>(\alpha)</td>
<td>(\epsilon)</td>
<td>(\alpha \beta)</td>
<td>(\beta)</td>
</tr>
<tr>
<td>(\beta)</td>
<td>(\beta)</td>
<td>(\beta)</td>
<td>(\epsilon)</td>
<td>(\alpha)</td>
</tr>
<tr>
<td>(\alpha \beta)</td>
<td>(\alpha \beta)</td>
<td>(\beta)</td>
<td>(\alpha)</td>
<td>(\epsilon)</td>
</tr>
</tbody>
</table>

Chapter 5. Problem 46. Show that for \(n \geq 3\), \(Z(S_n) = \{\epsilon\}\).

**Solution 1:** Suppose \(\sigma \in Z(S_{n+1})\), and let \(\alpha\) be some permutation in \(S_n\). Then we can think of \(\alpha\) in \(S_{n+1}\) as the permutation which does the same thing to the first \(n\) letters, and fixes the last one. So, \(\sigma \alpha = \alpha \sigma\) must hold \(\forall \alpha \in S_n\).

Let \(\beta = (123\ldots n)\), and suppose \(\sigma\) sends \((n+1)\) to \(k \neq n+1\) (we’ll write this as \(\sigma(n+1) = k\)). Then \(\beta \sigma\) sends \((n+1)\) to \(k+1\) \((\beta \sigma(n+1) = k+1)\), and \(\sigma \beta\) sends \((n+1)\) to \(k\), since \(\beta\) fixes \((n+1)\) \((\beta \sigma(n+1) = \sigma(n+1) = k)\), which must be equal to \(k+1\), since \(\sigma \beta = \beta \sigma\) since \(\sigma \in Z(S_{n+1})\). But \(k \neq k+1\), hence \(\sigma\) must actually fix \((n+1)\), that is \(\sigma(n+1) = n+1\). Since \(\sigma\) was arbitrarily chosen from \(Z(S_{n+1})\), we see that, in fact, every element of \(Z(S_{n+1})\) must fix \((n+1)\). This, and our initial observation that these elements must commute with each permutation in \(S_n\), shows that, in fact, \(Z(S_{n+1}) \subseteq Z(S_n)\).

This creates a chain of containment, and we know that \(Z(S_3) \supseteq Z(S_n)\) for any \(n \geq 3\). But upon examination, we see that the only element in \(Z(S_3)\) is the identity, \(\epsilon\), which we know to be contained in the center of every group. Hence, \(Z(S_n) = \{\epsilon\} \ \forall n \geq 3\).

**Solution 2:** Suppose \(\sigma \in S_n\) is not the identity element. We claim that \(\sigma \notin Z(S_n) = \{\alpha \in S_n : \alpha \beta = \beta \alpha \ \forall \beta \in S_n\}\). To prove this, we need only find one \(\tau \in S_n\) for which \(\sigma \tau \neq \tau \sigma\). As \(\sigma\) is not the identity, there is a number \(a\) such that \(\sigma(a) \neq a\). Set \(b = \sigma(a)\). Since \(n \geq 3\), there is a distinct number \(c\) such that \(c \neq a\) and \(c \neq b\). Set \(\tau = (ac)\), the 2-cycle that exchanges \(a\) and \(c\) and fixes everything else (including \(b\)). Now consider

\[
\begin{align*}
(\sigma \tau)(a) &= \sigma(\tau(a)) = \sigma(c) \\
(\tau \sigma)(a) &= \tau(\sigma(a)) = \tau(b) = b = \sigma(a).
\end{align*}
\]

Since \(\sigma \in S_n\), \(\sigma\) is one-to-one, so if \(\sigma \tau = \tau \sigma\), then we must have

\[
\sigma(c) = (\sigma \tau)(a) = (\tau \sigma)(a) = \sigma(a),
\]

so that \(c = a\). This is a contradiction, since we chose \(c\) such that \(c \neq a\). Thus \(\sigma(c) \neq \sigma(a)\), so we are forced to conclude that \(\sigma \tau \neq \tau \sigma\). Hence, given \(\sigma \neq \epsilon\) (where \(\epsilon\) denotes the identity permutation in \(S_n\)), there exists a 2-cycle \(\tau \in S_n\) such that \(\sigma \tau \neq \tau \sigma\). Therefore, we conclude that \(\sigma \notin Z(S_n)\), so \(Z(S_n) = \langle \epsilon \rangle\) is the trivial subgroup of \(S_n\).