1. (20 points) Consider the function

\[ f(x, y) = 2xy - 3y^2 \]

near the point \( P(5, 5) \).

(a) Find the rate of change of \( f(x, y) \) at \( P \) in the direction of the vector \( \mathbf{v} = \langle 4, 3 \rangle \).

\textbf{Solution:} The rates of change of \( f \) at \( P \) are measured by the directional derivatives, \( D_{\mathbf{u}} f(P) \). Now \( D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} \), where

\[ \nabla f(P) = \langle f_x(P), f_y(P) \rangle = \langle 2y|_{(5,5)}, 2x - 6y|_{(5,5)} \rangle = \langle 10, -20 \rangle. \]

Now, we need to find the unit vector \( \mathbf{u} \) in the direction of \( \mathbf{v} = \langle 4, 3 \rangle \), which is

\[ \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 4, 3 \rangle}{\sqrt{4^2 + 3^2}} = \langle \frac{4}{5}, \frac{3}{5} \rangle. \]

Thus, the rate of change of \( f(x, y) \) at \( P \) in the specified direction is

\[ D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} = \langle 10, -20 \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle = (10)(4/5) + (-20)(3/5) = 8 - 12 = -4. \]

(b) In which direction does the function \( f(x, y) \) increase most rapidly at \( P \)?

\textbf{Solution:} The directional derivative of \( f \) at \( P \) will be largest when \( \nabla f(P) \) and \( \mathbf{u} \) point in the same direction, since \( D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} = |\nabla f(P)||\mathbf{u}| \cos \theta = |\nabla f(P)| \cos \theta \), where \( \theta \) is the angle between \( \nabla f(P) \) and \( \mathbf{u} \). Now \( \nabla f(P) = \langle 10, -20 \rangle \), so we want \( \mathbf{u} \) to be the unit vector in this direction, i.e.,

\[ \mathbf{u} = \langle \frac{10}{\sqrt{10^2 + (-20)^2}}, \frac{-20}{\sqrt{10^2 + (-20)^2}} \rangle = \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle. \]

(c) Is there a direction \( \mathbf{u} \) in which the rate of change of \( f(x, y) \) at \( P \) equals -25? Give reasons for your answer.

\textbf{Solution:} We know that \( -|\nabla f(P)| \leq D_{\mathbf{u}} f(P) \leq |\nabla f(P)| \) for all directions \( \mathbf{u} \), so let’s compute

\[ |\nabla f(P)| = |\langle 10, -20 \rangle| = \sqrt{10^2 + (-20)^2} = 10\sqrt{5} \approx 22.36. \]

Thus, it is impossible to find a direction in which the rate of change of \( f(x, y) \) at \( P \) is equal to -25 since \( -25 < -|\nabla f(P)| \approx -22.36 \).

2. (30 points) Consider the surfaces \( S_1 \) and \( S_2 \), given by the equations

\[ S_1 : x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0 \quad \text{and} \quad S_2 : x^2 + y^2 + z^2 = 11, \]

and the point \( P(1, 1, 3) \), which is on both surfaces.

(a) Find the equation of the tangent plane to \( S_1 \) at \( P \).

\textbf{Solution:} To know the equation of the tangent plane to \( S_1 \) at \( P \), we need a point (i.e., \( P \) itself) and a vector perpendicular to \( S_1 \) at \( P \). Here’s where the gradient comes in, as \( \nabla f \)
is perpendicular to the level surface of the function \( f(x, y, z) = x^3 + 3x^2y^2 + y^3 + 4xy - z^2 \) (for \( S_1 \) is given by \( f(x, y, z) = 0 \)). Now we calculate

\[
\nabla f = \langle 3x^2 + 6xy^2 + 4y, 6x^2y + 3y^2 + 4x, -2z \rangle
\]

so

\[
\]

Hence the equation for the tangent plane, \( T_1 \), to \( S_1 \) at \( P \) is

\[
[13](x - 1) + [13](y - 1) + [-6](z - 3) = 0 \quad \text{or} \quad 13x + 13y - 6z = 8.
\]

(b) Find the equation of the tangent plane to \( S_2 \) at \( P \).

**Solution:** As above, the equation for the tangent plane to \( S_2 \) at \( P \) requires a point (i.e., \( P \) itself) and a vector perpendicular to \( S_2 \) at \( P \). Here's where the gradient comes in, as \( \nabla g \) is perpendicular to the level surface of the function \( g(x, y, z) = x^2 + y^2 + z^2 \) (for \( S_2 \) is given by \( g(x, y, z) = 11 \)). Now we calculate

\[
\nabla g = \langle 2x, 2y, 2z \rangle \quad \text{so} \quad \nabla g(P) = \langle 2[1], 2[1], 2[3] \rangle = \langle 2, 2, 6 \rangle.
\]

Hence the equation for the tangent plane, \( T_2 \), to \( S_2 \) at \( P \) is

\[
[2](x - 1) + [2](y - 1) + [6](z - 3) = 0 \quad \text{or} \quad x + y + 3z = 11.
\]

(c) Find the *parametric equations* for the line of intersection of the tangent planes in parts (a) and (b).

**Solution:** To find the line tangent to the curve of intersection of \( S_1 \) and \( S_2 \) at \( P \), which must then be tangent to \( S_1 \) at \( P \) as well as to \( S_2 \) at \( P \), we can quickly see that what we really are after is the line of intersection of the two planes from parts (b) and (c) above. So we need a direction for this line and a point on it. By definition, essentially, \( P \) is that point, and the direction will have to be a direction that is both in \( T_1 \) and in \( T_2 \), so it is perpendicular to both the normal vector \( \mathbf{n}_1 = \langle 13, 13, -6 \rangle \) to \( T_1 \) and to the normal vector \( \mathbf{n}_2 = \langle 2, 2, 6 \rangle \) to \( T_2 \). We find such a vector using the *cross product*,

\[
\mathbf{n}_1 \times \mathbf{n}_2 = \langle 13, 13, -6 \rangle \times \langle 2, 2, 6 \rangle = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
13 & 13 & -6 \\
2 & 2 & 6 \\
\end{vmatrix} = \langle 90, -90, 0 \rangle.
\]

Thus, the parametric equations for the line tangent to both surfaces \( S_1 \) and \( S_2 \) at the point \( P \) are

\[
x = 1 + 90t, \quad y = 1 - 90t, \quad z = 3.
\]

[**Note:** This is the line tangent to the curve of intersection of the surfaces, \( S_1 \) and \( S_2 \), at the point \( P \).]

3. (10 points) Find the limit of \( f \) as \( (x, y) \to (0, 0) \) or show that the limit does not exist:

\[
f(x, y) = \frac{y^2}{x^2 + y^2}
\]
Solution: Let’s rewrite the function using polar coordinates, \( r \) and \( \theta \), and recognize that this limit will correspond to letting \( r \to 0 \). Now
\[
\frac{y^2}{x^2 + y^2} = \frac{(r \sin \theta)^2}{r^2} = \frac{r^2 \sin^2 \theta}{r^2} = \sin^2 \theta,
\]
which clearly depends on the value of \( \theta \), even as \( r \to 0 \). So the limit does not exist.

We can also show this by comparing the value of the limit along two different lines, \( y = m_1 x \) and \( y = m_2 x \) for different slopes, \( m_1 \) and \( m_2 \). For efficiency, let’s consider all lines \( y = mx \), so \((x, y) \to (0, 0)\) along \( y = mx \) is accomplished by letting \( x \to 0 \) and evaluating
\[
\lim_{x \to 0} \frac{(mx)^2}{x^2 + (mx)^2} = \lim_{x \to 0} \frac{m^2 x^2}{x^2(1 + m^2)} = \lim_{x \to 0} \frac{m^2}{1 + m^2} = \frac{m^2}{1 + m^2}.
\]
So, if \( m_1 = 0 \) and we let \((x, y) \to (0, 0)\) along the line \( y = 0 \), the projected limit is \( \frac{(0)^2}{1 + (0)^2} = 0 \). However, when \( m_2 = 1 \) and \((x, y) \to (0, 0)\) along the line \( y = x \), we find that the projected value of the limit is \( \frac{(1)^2}{1 + (1)^2} = \frac{1}{2} \). Since these differ, we conclude that the limit does not exist.

4. (25 points) Find all the local maximum, local minimum, and saddle points of the function
\[
f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy.
\]

Solution: We begin by finding the partial derivatives of \( f \):
\[
f_x(x, y) = 12x - 6x^2 + 6y \quad \text{and} \quad f_y(x, y) = 6y + 6x.
\]
Setting \( f_y = 0 \), we find that \( 6y = -6x \), which we can now plug in to the equation \( f_x = 0 \) to get
\[
12x - 6x^2 + [-6x] = 6x - 6x^2 = 6x(1 - x) = 0
\]
so \( x = 0 \) or \( x = 1 \). When \( x = 0 \), the equation \( 6y = -6x \) tells us \( y = 0 \) as well so we find the critical point \((0, 0)\) of \( f \). When \( x = 1 \), \( 6y = -6x \) implies that \( y = -1 \) and \((1, -1)\) is another critical point of \( f \).

To test these points, we compute the second order partial derivatives,
\[
f_{xx}(x, y) = 12 - 12x, \quad \text{and} \quad f_{xy}(x, y) = 6 = f_{yx}(x, y), \quad f_{yy}(x, y) = 6.
\]
Therefore, the discriminant is
\[
D_f(x, y) = [f_{xx}][f_{yy}] - [f_{xy}]^2 = [12 - 12x][6] - [6]^2 = 72 - 72x - 36 = 36 - 72x.
\]

We use this to test our first critical point:
\[
D_f(0, 0) = 36 - 72[0] = 36 > 0 \quad \text{and} \quad f_{xx}(0, 0) = 12 - 12[0] = 12 > 0
\]
implies
\[
f(0, 0) = 0 \text{ is a local minimum value of } f.
\]
Now, checking our other critical point, we find:
\[
D_f(1, -1) = 36 - 72[1] = -36 < 0
\]
implies
\[
f(1, -1) = 1 \text{ is a saddle point of } f.
\]
5. (40 points) Consider the function
\[ f(x, y) = 2x^2 - 4x + y^2 - 4y + 1 \]
on the closed triangular region \( R \) bounded by the lines \( x = 0, \ y = 3 \) and \( y = x \).

(a) Find the absolute maximum and minimum values of \( f(x, y) \) on \( R \).

**Solution:** Let’s begin by drawing the region \( R \) described in the problem.

So our strategy will be to first locate local extrema, then to test each of the three boundary lines using Calculus I techniques.

First, let’s identify the critical points of \( f \). So we compute

\[ f_x(x, y) = 4x - 4 \quad \text{and} \quad f_y(x, y) = 2y - 4. \]

Now \( f_x(x, y) = 4x - 4 = 0 \) only when \( x = 1 \), and \( f_y(x, y) = 2y - 4 = 0 \) only when \( y = 2 \), so \((1, 2)\) is the only critical point of the function \( f \). Since we’re looking for *absolute extrema*, we don’t need to employ any Second Derivative Tests, but content ourselves to evaluate


With the local extrema now determined, we restrict our attention to the boundaries. So consider

\[ f(0, y) = y^2 - 4y + 1 \implies f'(0, y) = 2y - 4 = 0 \quad \text{when} \quad y = 2. \]

So, on the boundary \( x = 0 \), the “critical point” is at \((0, 2)\), so we record the value of at this point and the endpoints of this edge:

\[ f(0, 0) = 1 \quad \text{and} \quad f(0, 2) = -3 \quad \text{and} \quad f(0, 3) = -2. \]

Next, along the edge \( y = 3 \) we have

\[ f(x, 3) = 2x^2 - 4x - 2 \implies f'(x, 3) = 4x - 4 = 0 \quad \text{when} \quad x = 1. \]

Hence, when \( y = 3 \), the “critical point” is at \((1, 3)\), so we record the values

\[ f(0, 3) = -2 \quad \text{and} \quad f(1, 3) = -4 \quad \text{and} \quad f(3, 3) = 4. \]

Finally, on the edge \( y = x \), we have

\[ f(x, x) = 2x^2 - 4x + [x]^2 - 4[x] + 1 = 3x^2 - 8x + 1 \implies f'(x, x) = 6x - 8 = 0 \quad \text{when} \quad x = 4/3. \]
So, on $y = x$, the only “critical point” is at $(4/3, 4/3)$, so we record

$$f(0, 0) = 1 \quad \text{and} \quad f(4/3, 4/3) = -\frac{13}{3} \quad \text{and} \quad f(3, 3) = 4.$$ 

Therefore, the absolute maximum value of $f(x, y)$ is $f(3, 3) = 4$ and the absolute minimum value of $f(x, y)$ is $f(1, 2) = -5$.

(b) Find the average value of $f(x, y)$ on $\mathcal{R}$.

**Solution:** The average value of $f(x, y)$ is

$$f_{\text{ave}} = \frac{1}{\text{Area}(\mathcal{R})} \iint_{\mathcal{R}} f(x, y) \, dA.$$ 

While we could computer $\text{Area}(\mathcal{R})$ as a double integral, it is clearly going to be easier to find this area by observing that $\mathcal{R}$ is a triangle of base $b = 3$ and height $h = 3$, so its area is

$$\text{Area}(\mathcal{R}) = \frac{1}{2}bh = \frac{1}{2}(3)(3) = \frac{9}{2}.$$ 

Therefore, the average value of $f(x, y)$ on the region $\mathcal{R}$ is

$$f_{\text{ave}} = \frac{1}{\text{Area}(\mathcal{R})} \iint_{\mathcal{R}} f(x, y) \, dA = \frac{1}{\frac{9}{2}} \iint_{\mathcal{R}} f(x, y) \, dA = \frac{2}{9} \iint_{\mathcal{R}} [2x^2 - 4x + y^2 - 4y + 1] \, dA$$

$$= \frac{2}{9} \int_{y=0}^{3} \int_{x=0}^{y} [2x^2 - 4x + y^2 - 4y + 1] \, dx \, dy$$

$$= \frac{2}{9} \int_{y=0}^{3} \left[ \frac{2x^3}{3} - 2x^2 + xy^2 - 4xy + x \right]_{x=0}^{y} \, dy$$

$$= \frac{2}{9} \int_{y=0}^{3} \left[ \frac{2}{3}y^3 - 2y^2 + y^3 - 4y^2 + y \right] - (0) \, dy$$

$$= \frac{2}{9} \int_{y=0}^{3} \left[ \frac{2}{3}y^3 - 2y^2 + y^3 - 4y^2 + y \right] \, dy$$

$$= \frac{2}{9} \left[ \frac{2}{3} \cdot \frac{3}{4}y^4 - \frac{2}{3} \cdot \frac{3}{4}y^3 + \frac{3}{4}y^4 - 4 \cdot \frac{3}{4}y^3 + \frac{3}{2}y^2 \right]_{y=0}$$

$$= \frac{2}{9} \left[ \left( \frac{81}{6} - \frac{54}{3} + \frac{81}{4} - \frac{108}{3} + \frac{9}{2} \right) - (0) \right]$$

$$= \frac{2}{9} \left[ \left( \frac{81}{6} - \frac{54}{3} + \frac{81}{4} - \frac{108}{3} + \frac{9}{2} \right) - (0) \right]$$

$$= \frac{2}{9} \left[ \frac{-189}{12} \right] = -\frac{21}{6} = -\frac{7}{2}.$$ 

6. (25 points) **Improper Double Integrals.** Consider the integral

$$\int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy.$$ 

(a) First, sketch the region over which the integral is being taken.
While I can’t (and don’t expect you to do so, either) draw the entire first quadrant, that is the region over which the integral is being taken and which, I hope, you understand the illustration above to indicate.

(b) Next, change the integral into an equivalent integral using polar coordinates, \( r \) and \( \theta \).

**Solution:** Recall, first, that \( r^2 = x^2 + y^2 \) and that \( dx
dy = dA = r
dr
d\theta \). Next, the polar coordinates of points in the first quadrant are of the form \((r, \theta)\) with \( 0 \leq r < \infty \) and \( 0 \leq \theta \leq \frac{\pi}{2} \), so the integral we are given is equivalent to the following polar integral:

\[
\int_{y=0}^{\infty} \int_{x=0}^{\infty} \frac{1}{(1 + x^2 + y^2)^2} \; dx \; dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} \frac{1}{(1 + r^2)^2} \; r \; dr \; d\theta.
\]

(c) Finally, evaluate the polar integral in part (b).

**Solution:** Starting where we left off in part (b), we have

\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{(1 + x^2 + y^2)^2} \; dx \; dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} \frac{1}{(1 + r^2)^2} \; r \; dr \; d\theta
\]

\[
= \int_{\theta=0}^{\pi/2} \left[ \lim_{b \to \infty} \int_{r=0}^{b} \frac{r}{(1 + r^2)^2} \; dr \right] \; d\theta
\]

\[
= \int_{\theta=0}^{\pi/2} \left[ \lim_{b \to \infty} \frac{1}{2} (1 + r^2)^{-1} \right]_{r=0}^{b} \; d\theta
\]

\[
= \int_{\theta=0}^{\pi/2} \left[ \lim_{b \to \infty} \left( \frac{-1}{2(1+b^2)} - \frac{-1}{2(1+0^2)} \right) \right] \; d\theta
\]

\[
= \int_{\theta=0}^{\pi/2} \left[ \frac{1}{2} \right] \; d\theta
\]

\[
= \left[ \frac{1}{2} \theta \right]_{\theta=0}^{\pi/2}
\]

\[
= \frac{\pi}{4}.
\]