1. **Calculus III Exam 1 Practice Problems - Solutions**

1. Compute the following:

   (a) \[ \lim_{t \to 1} \left[ (1 - t^2)\vec{i} + (te^t)\vec{j} + (\sin \pi t)\vec{k} \right] = \lim_{t \to 1} (1 - t^2, te^t, \sin \pi t) = (1, 1, \sin \pi) = (0, e, 0) \]

   (b) \[ \int_0^1 \left[ \cos t \vec{j} + (1 + t^2)\vec{j} - \vec{k} \right] \, dt = \int_0^1 \cos t \, dt, \int_0^1 (1 + t^2)dt, \int_0^1 -dt = \langle [\sin t]_0^1, [\tan^{-1} t]_0^1, [-t]_0^1 \rangle = \langle \sin 1, \pi/4, -1 \rangle \]

   (c) \[ \int \left[ (\ln t)\vec{i} + (\sinh t - \cosh t)\vec{j} \right] \, dt = \int (\ln t) \, dt, \int (\sinh t - \cosh t) \, dt, \int 0 \, dt = \langle t \ln t - t + C_1, \cosh t - \sinh t + C_2, C_3 \rangle \]

   (d) \[ \lim_{t \to 0} \left[ (\tan^{-1} t)\vec{j} - (t^{-3/2})\vec{k} \right] = \lim_{t \to 0} \left[ 0, \lim_{t \to 0} \tan^{-1} t, \lim_{t \to 0} -t^{-3/2} \right] = (0, \pi/2, 0) \]

   (e) \[ \frac{d^3}{dt^3}(t^2, t^3) = \frac{d}{dt}(\frac{d}{dt}(t^2), \frac{d}{dt}(t^3)) = \frac{d}{dt}(1, 2t, 3t^2) = \langle \frac{d}{dt}[1], \frac{d}{dt}[2t], \frac{d}{dt}[3t^2] \rangle = (0, 2, 6t) \]

   (f) \[ \frac{d}{dt} \left[ e^{-t} \cos t\vec{i} + e^{-t} \sin t\vec{j} + \ln t\vec{k} \right] = \langle \frac{d}{dt}[e^{-t} \cos t], \frac{d}{dt}[e^{-t} \sin t], \frac{d}{dt}[\ln t] \rangle = \langle e^{-t}(-\sin t) + (-e^{-t}) \cos t, e^{-t} \cos t + (-e^{-t}) \sin t, 1/t \rangle \]

   (g) \[ \frac{d}{dt} \left[ (5 - t^2)\vec{i} + (t \sin t)\vec{j} - e^{-t}\vec{k} \right] = \langle \frac{d}{dt}[5 - t^2], \frac{d}{dt}[t \sin t], \frac{d}{dt}[-e^{-t}] \rangle = \langle -2t, \sin t + t \cos t, -e^{-t} \rangle \]

   (h) \[ \frac{d}{dt} \left[ \left( \tan t \right)\vec{i} - \left( t \sin t \right)\vec{j} - \left( t \cos t \right)\vec{k} \right] = \langle \frac{d}{dt}[\tan t], \frac{d}{dt}[t \sin t], \frac{d}{dt}[t \cos t] \rangle = \langle (\sec^2 t, 0), (\sec^2 t, t), (\sec^2 t, -t) \rangle \]

   (i) \[ \frac{d}{dt} \left[ \left( e^t \right)\vec{j} \right] = \langle \frac{d}{dt}[0], \frac{d}{dt}[e^t], \frac{d}{dt}[0] \rangle \times \langle 0, 0, -t^2 \rangle = \langle 0, 0, -t^2 \rangle \times \langle 0, e^t, 0 \rangle = \langle 0, 2te^t, 0 \rangle \times \langle 0, 0, -t^2 \rangle = \langle -2te^t, 0, 0 \rangle \}

   (j) \[ \lim_{t \to 0} \left\langle \frac{1 - \cos t}{t^3}, e^{-1/t^2} \right\rangle = \lim_{t \to 0} \frac{1 - \cos t}{t^3}, \lim_{t \to 0} t^3, \lim_{t \to 0} e^{-1/t^2} = (0, 0, 0) \]

   (k) \[ \int_0^{\pi/4} \left[ \cos 2t\vec{i} + \sin 2t\vec{j} + t \sin t\vec{k} \right] \, dt = \left\langle \int_0^{\pi/4} \cos 2t \, dt, \int_0^{\pi/4} \sin 2t \, dt, \int_0^{\pi/4} t \sin t \, dt \right\rangle = \left\langle \left\{ \frac{1}{2} \sin 2t \right\}_{t=0}^{\pi/4}, \left\{ -\frac{1}{2} \cos 2t \right\}_{t=0}^{\pi/4}, \left\{ -t \cos t + \sin t \right\}_{t=0}^{\pi/4} \right\rangle = (1/2, 1/2, -\pi/8 + \pi/2) \]

   (l) \[ \int_1^{\sqrt{2}} \left[ \sqrt{t} + te^{-t}\vec{j} + \frac{1}{t^2}\vec{k} \right] \, dt = \left\langle \int_1^{\sqrt{2}} \sqrt{t} \, dt, \int_1^{\sqrt{2}} te^{-t} \, dt, \int_1^{\sqrt{2}} -\frac{1}{t^2} \, dt \right\rangle = \left\langle \left\{ \frac{2}{3}t^{3/2} \right\}_{t=1}^{\sqrt{2}}, \left\{ -te^{-t} \right\}_{t=1}^{\sqrt{2}}, \left\{ -t^{-1} \right\}_{t=1}^{\sqrt{2}} \right\rangle = \left\langle \frac{14}{3}, -5e^{-1} + 2e^{-1}, \frac{3}{4} \right\rangle \]
2. Find the unit tangent vector and unit normal vector to the space curve
\[ \mathbf{r}(t) = (t^3 - 1)\mathbf{i} - \left(\frac{3\sqrt{2}t}{2} - t^2\right)\mathbf{j} + (3t)\mathbf{k} \]
at the point where \( t = 1 \).

**Solution:** First of all, \( \mathbf{r}'(t) = (3t^2, -3\sqrt{2}t, 3) \), so
\[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{(3t^2, -3\sqrt{2}t, 3)}{\sqrt{(9t^4) + (18t^2) + (9)}} = \frac{(t^2, -\sqrt{2}t, 1)}{t^2 + 1} = \left\langle \frac{t^2}{t^2 + 1}, \frac{\sqrt{2}t}{t^2 + 1}, \frac{1}{t^2 + 1} \right\rangle \]
and
\[ \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \frac{\langle (t^2 + 1)^{-2}2t, -2[t(t^2+1)^{-2}(2t) + (t^2+1)^{-1}], -(t^2+1)^{-2}(2t) \rangle}{\sqrt{[2t(t^2+1)^{-2}]^2 + [2(-2t(t^2+1)^{-2})+(t^2+1)^{-1}]^2 + [-2t(t^2+1)^{-2}]^2}} \]

3. If \( \mathbf{r} = (x, y, z), \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \), show that the vector equation \( (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0 \) represents a sphere. Find its center and radius.

**Solution:** Consider \( \langle x - a_1, y - a_2, z - a_3 \rangle \cdot \langle x - b_1, y - b_2, z - b_3 \rangle = (x-a_1)(x-b_1) + (y-a_2)(y-b_2) + (z-a_3)(z-b_3) = (x^2 - (a_1+b_1)x) + (y^2 - (a_2+b_2)y) + (z^2 - (a_3+b_3)z) + a_1b_1 + a_2b_2 + a_3b_3 \)
Then \( (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0 \) is equivalent to the equation
\[ \left(\frac{x - a_1 + b_2}{2}\right)^2 + \left(\frac{y - a_2 + b_2}{2}\right)^2 + \left(\frac{z - a_3 + b_3}{2}\right)^2 = \frac{1}{2}(a - b) \]
This is a sphere centered at \( C(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}) = \frac{1}{2}(\vec{a} + \vec{b}) \) (the midpoint between the points \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\)) and having radius \( \sqrt{\frac{1}{2}(\vec{a} - \vec{b})} \), (one half the distance between the points \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\)).

4. Find the length of the space curve \( \mathbf{r}(t) = \left(\frac{t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{t^3}{6} + \frac{1}{2t}\right)\mathbf{j} - \left(\frac{t}{\sqrt{2}}\right)\mathbf{k} \), over the interval \( 1 \leq t \leq 2 \).

**Solution:** The arc length of the curve \( \mathbf{r}(t) \) is \( L = \int_1^2 ||\mathbf{r}'(t)|| dt = \int_1^2 \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{t^3}{6} + \frac{1}{2t}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2}dt = \int_1^2 \sqrt{\frac{1}{2} + \frac{t^4}{4} + \frac{1}{4t^2} + \frac{1}{2t}}dt = \int_1^2 \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2}dt = \int_1^2 \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t}\right)^2}dt = \int_1^2 \frac{t^2}{2} + \frac{1}{2t}dt = \left[\frac{t^3}{6} + \frac{1}{2}\right]_1^2 = \frac{8}{6} - \frac{1}{2} = \frac{1}{3} \]

5. Show that if \( \mathbf{r} \) is a vector function such that \( \mathbf{r}'' = (\mathbf{r}')' \) exists, then \( \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) \).

**Solution:** Let \( \mathbf{r}(t) \) be a vector-valued function and assume \( \mathbf{r}''(t) \) exists. Then \( \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \frac{d}{dt}[\mathbf{r}'(t)] + \frac{d}{dt}[\mathbf{r}(t)] \times \mathbf{r}'(t) = \mathbf{r}(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{r}(t) \times \mathbf{r}''(t) \), since \( \mathbf{v} \times \mathbf{v} = \mathbf{0} \) for all vectors \( \mathbf{v} \).
6. Let \( L_1, L_2 \) and \( L_3 \) be the three lines given below

\[
\begin{align*}
x &= -s, & y &= 4 + 2s, & z &= 6 + 3s \\
x &= 2 + t, & y &= 4 + 2t, & z &= 6 - 3t \\
x &= 2 + u, & y &= -2u, & z &= 6 + 3u
\end{align*}
\]

(a) Two of these lines intersect. Which two? At which point?

**Solution:** Let’s first see if \( L_1 \) and \( L_2 \) intersect. If so, there must be some value \( s_0 \) of the parameter \( s \) of \( L_1 \) and \( t_0 \) of the parameter \( t \) of \( L_2 \) such that

\[
\begin{align*}
-s_0 &= 2 + t_0 \\
4 + 2s_0 &= 4 + 2t_0 \\
6 + 3s_0 &= 6 - 3t_0
\end{align*}
\]

The second of these equations requires that \( s_0 = t_0 \) (subtract 4 from both sides and divide by 2), but the third equation requires that \( s_0 = -t_0 \) (subtract 6 from both sides and divide by 3). Thus \( s_0 = t_0 = 0 \), which can’t satisfy the first equation. Hence, line \( L_1 \) and \( L_2 \) do not intersect.

If \( L_1 \) and \( L_3 \) intersect, then there are values \( s_0 \) and \( u_0 \) satisfying the system of equations

\[
\begin{align*}
-s_0 &= 2 + u_0 \\
4 + 2s_0 &= -2u_0 \\
6 + 3s_0 &= 6 + 3u_0
\end{align*}
\]

The third equation above requires \( s_0 = u_0 \), so the first equation now implies that \( s_0 = u_0 = -1 \). Checking this against the second equation, we find the left-hand side is \( 4 + 2(-1) = 2 \) and the right-hand side is \( -2(-1) = 2 \), so all three equations are satisfied when \( s_0 = u_0 = -1 \). Thus lines \( L_1 \) and \( L_3 \) intersect at the point where \( s_0 = u_0 = -1 \), i.e., at the point \((1, 2, 3)\) (using \( s_0 = -1 \) in the equation for line \( L_1 \)).

It also appears that lines \( L_2 \) and \( L_3 \) do not intersect, for otherwise \( t_0 = u_0 \) (from equation 1) and \(-t_0 = u_0 \) (from equation 3), so that \( t_0 = u_0 = 0 \), but then \( 4 = 0 \) (from equation 2), which is a contradiction.

(b) Another pair is skew. How do you know this pair to be skew?

**Solution:** We’ve seen that lines \( L_1 \) and \( L_2 \) do not intersect and that the lines \( L_2 \) and \( L_3 \) do not intersect. Two lines in space that do not intersect are either parallel or they are skew. To be parallel, the direction vectors of the lines must be scalar multiples of one another. Now the direction vector of line \( L_1 \) is \( \vec{d}_1 = (-1, 2, 3) \), the direction vector of line \( L_2 \) is \( \vec{d}_2 = (1, 2, -3) \), and the direction vector of line \( L_3 \) is \( \vec{d}_3 = (1, -2, 3) \). We can see that no two of these lines are parallel, for there are no non-zero scalars \( \lambda \) such that \( \vec{d}_1 = \lambda \vec{d}_2 \) or \( \vec{d}_1 = \lambda \vec{d}_3 \) or \( \vec{d}_2 = \lambda \vec{d}_3 \). Thus, both the pair \( L_1, L_2 \) and the pair \( L_2, L_3 \) are skew lines.

(c) Find the distance between the two skew lines that you have just identified. (One way to do this is to find a pair of parallel planes, each of which contains one of the lines in your skew pair, and then to find the distance between these parallel planes.)

**Solution:** The distance between the lines \( L_1 \) and \( L_2 \) will be the distance between the parallel planes \( M \) and \( N \) through the points \( P(0, 4, 3) \) (the point on \( L_1 \) corresponding to \( s = 0 \)) and \( Q(2, 4, 6) \) (the point on \( L_2 \) corresponding to \( t = 0 \)), respectively, and each perpendicular to the vector \( \vec{n} = \vec{d}_1 \times \vec{d}_2 \), as then \( L_1 \) will lie in \( M \) and \( L_2 \) will lie in \( N \). Now

\[
\vec{n} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 2 & 3 \\
1 & 2 & -3
\end{vmatrix} = [(2)(-3)-(2)(3)]\vec{i} - [(1)(-3)-(1)(3)]\vec{j} + [(1)(2)-(1)(2)]\vec{k} = \langle -12, 0, -4 \rangle.
\]
Hence $M$ has equation $-12(x - 0) + 0(y - 4) + -4(z - 3) = -12x - 4z + 12 = 0$ or $3x + z = 3$, while $N$ has equation $-12(x - 2) + 0(y - 4) + -4(z - 6) = -12x - 4z + 48 = 0$ or $3x + z = 12$. Then the distance between $M$ and $N$, and hence between the lines $L_1$ and $L_2$, is

$$D = ||\text{proj}_nPQ|| = \frac{|(-12,0,-4) \cdot (2,0,3)|}{||(2,0,-3)||} = \frac{|-24 + 0 - 12|}{\sqrt{144 + 0 + 16}} = \frac{36}{160} = \frac{9}{10}$$

by Formula 8 of Section 9.5 in the book.

The distance between the lines $L_2$ and $L_3$ is the distance between the parallel planes $M'$ and $N'$ through the points $Q(2,4,6)$ (the point on $L_2$ corresponding to $t = 0$) and $R(2,0,6)$ (the point on $L_3$ corresponding to $u = 0$). In order for $M'$ and $N'$ to be parallel and for them to contain $L_2$ and $L_3$, respectively, they must both be normal to the vector $m = d_2 \times d_3$:

$$m = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 1 & -2 & 3 \end{vmatrix} = [(2)(3)-(-2)(-3)]\hat{i} - [(1)(3)-(1)(-3)]\hat{j} + [(1)(-2)-(1)(2)]\hat{k} = (0,-6,-4).$$

Hence $M'$ has equation $[0][x - 2] + [-6][y - 4] + [-4][z - 6] = -6y - 4z + 48 = 0$ or $3y + 2z = 24$, while $N'$ has equation $[0][x - 2] + [-6][y - 0] + [-4][z - 6] = -6y - 4z + 24 = 0$ or $3y + 2z = 12$. Then the distance between $M'$ and $N'$, and hence between the lines $L_2$ and $L_3$, is

$$D = ||\text{proj}_mQR|| = \frac{|(0,-6,-4) \cdot (0,-4,0)|}{||(0,-6,-4)||} = \frac{|0 + 24 + 0|}{\sqrt{0 + 36 + 16}} = \frac{24}{\sqrt{52}} = \frac{12}{\sqrt{13}}$$

by Formula 8 of Section 9.5 in the book.

7. We obtain a vector normal to a path specified by $r(t)$ by differentiating the unit tangent vector and normalizing it. In this problem, you will show that this is a reasonable thing to do.

(a) First prove that $r'(t)$ is always orthogonal to $r(t)$ when $r(t)$ is of constant magnitude. 

Solution: Suppose $r(t)$ has constant magnitude $||r(t)|| = k$ for some constant $k$. Then, squaring both sides and using the identity $||v||^2 = v \cdot v$ for any vector $v$, we have the equation

$$r(t) \cdot r(t) = k^2$$

is a constant. Thus, differentiating both sides with respect to $t$ yields

$$r'(t) \cdot r(t) + r(t) \cdot r'(t) = 0,$$

so $2r(t) \cdot r'(t) = 0$, in which case $r(t) \cdot r'(t) = 0$. Therefore, $r(t) \perp r'(t)$ for all $t$, since $u \perp v$ if and only if $u \cdot v = 0$.

(b) Use this to explain why the vector $N = \frac{T'}{||T'||}$ is a unit vector which is normal to the curve specified by $r(t)$.

Solution: The vector $T(t) = \frac{r'(t)}{||r'(t)||}$ is the unit tangent vector to the curve specified by $r(t)$. That is, $T(t)$ is tangent to the curve and $||T(t)|| = 1$. Thus, as $T(t)$ is a vector function of constant length, it is always perpendicular to its derivative, $T'(t)$ by the previous part of this problem. Therefore, the vector $N(t) = \frac{T'(t)}{||T'(t)||}$ is a vector of length 1 in the same direction as $T'(t)$, so it is also perpendicular to $T(t)$, and hence is a unit vector which is normal to the curve $r(t)$. 
8. We define the \textit{curvature} of a path specified by \( \mathbf{r}(t) \) to be the magnitude of the vector \( \frac{d\mathbf{T}}{ds} \).

(a) Use a picture to illustrate why this is a reasonable definition.

\textbf{Solution:} Think about a circle. The parameter \( s \) measures the arc length, so the magnitude of the vector \( \frac{d\mathbf{T}}{ds} \) is measuring how quickly the tangent direction, \( \mathbf{T} \), changes with respect to the arc length as we traverse the circle.

(b) In spite of this natural definition, curvature is notoriously tedious to compute. Nevertheless, some straightforward formulas are available to make life a little more pleasant. One of these is

\[ \kappa = \frac{||\mathbf{r}' \times \mathbf{r}''||}{||\mathbf{r}'||^3} \]

Use the fact that \( \mathbf{a} = \left( \frac{d^2s}{dt^2} \right) \mathbf{T} + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N} \) to explain why this formula holds.

\textbf{Solution:} Recall that \( \mathbf{a} = \mathbf{r}'' \) and that \( \mathbf{T} = \frac{\mathbf{r}'}{||\mathbf{r}'||} \), so that \( \mathbf{T} \) and \( \mathbf{r}' \) are parallel vectors. We begin by considering the vector \( \mathbf{r}' \times \mathbf{r}'' = \mathbf{r}' \times \left( \left( \frac{d^2s}{dt^2} \right) \mathbf{T} + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N} \right) = \left( \frac{d^2s}{dt^2} \right) [\mathbf{r}' \times \mathbf{T}] + \kappa \left( \frac{ds}{dt} \right)^2 [\mathbf{r}' \times \mathbf{N}] = \left( \frac{d^2s}{dt^2} \right) \mathbf{B} \) since these vectors are parallel while \( \mathbf{r}' \times \mathbf{N} = ||\mathbf{r}'|| \mathbf{T} \times \mathbf{N} = ||\mathbf{r}'|| \mathbf{B} \). Thus, \( ||\mathbf{r}' \times \mathbf{r}''|| = ||\kappa \left( \frac{ds}{dt} \right)^3 \mathbf{B}|| = \kappa ||\mathbf{r}'||^3 \), since \( ||\mathbf{r}'|| \) is the speed, which is the same as \( \frac{ds}{dt} \). (Formally, \( s = \int_a^t ||\mathbf{r}'|| \, dt \), so the Fundamental Theorem of Calculus implies that \( \frac{ds}{dt} = ||\mathbf{r}'(t)|| \).) Therefore, solving for \( \kappa \), we obtain the formula

\[ \kappa = \frac{||\mathbf{r}' \times \mathbf{r}''||}{||\mathbf{r}'||^3} \]

(c) \textbf{BONUS:} Explain why \( \mathbf{r}' = \frac{ds}{dt} \mathbf{T} \) and then use this to explain why \( \mathbf{a} = \left( \frac{d^2s}{dt^2} \right) \mathbf{T} + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N} \).

\textbf{Solution:} As I have already shown above, \( \mathbf{r}' = ||\mathbf{r}'|| \mathbf{T} \) and \( ||\mathbf{r}'|| = \frac{ds}{dt} \) by the Fundamental Theorem of Calculus (my parenthetical remark near the end of the previous solution). Thus \( \mathbf{r}' = \frac{ds}{dt} \mathbf{T} \).

Thus, \( \mathbf{a} = \frac{d}{dt} [\mathbf{r}'(t)] = \frac{d}{dt} \left[ \frac{ds}{dt} \mathbf{T} \right] = \frac{d}{dt} \frac{ds}{dt} \cdot \mathbf{T} + \frac{ds}{dt} \frac{d}{dt} \mathbf{T} = \left( \frac{d^2s}{dt^2} \right) \mathbf{T} + \left( \frac{ds}{dt} \right) \left[ \frac{ds}{dt} \right] \mathbf{T} \), this last part by the Chain Rule. Now \( \frac{d\mathbf{T}}{ds} = \kappa \) by definition of curvature, so

\[ \mathbf{a} = \left( \frac{d^2s}{dt^2} \right) \mathbf{T} + \left( \frac{ds}{dt} \right)^2 \kappa \frac{\mathbf{T} \times \mathbf{N}}{||\mathbf{T} \times \mathbf{N}||} = \left( \frac{d^2s}{dt^2} \right) \mathbf{T} + \left( \frac{ds}{dt} \right)^2 \kappa \mathbf{N} = \left( \frac{d^2s}{dt^2} \right) \mathbf{T} + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N} \]

because \( \frac{d\mathbf{T}}{ds} \) is the unit vector in the direction of \( \mathbf{T}' \), which is \( \mathbf{N} \) by definition.
9. A particle that has been stationary at the origin \((0, 0, 0)\) suddenly begins to move in such a way that its acceleration at time \(t\) seconds is given by \(\mathbf{a}(t) = \langle 1, \cos t, \sin t \rangle\).

(a) Where will the particle be when \(t = \pi\) seconds?

**Solution:** The velocity function of the particle is \(\mathbf{v}(t) = \int \mathbf{a}(t) \, dt = \int \langle 1, \cos t, \sin t \rangle \, dt = (t + C_1, \sin t + C_2, -\cos t + C_3)\). Given the initial velocity \(\mathbf{v}(0) = \langle 0, 0, 0 \rangle\) (because the particle was stationary at the origin, its initial velocity is trivial), we solve for the unknown constants \(C_1, C_2, C_3\) by equating corresponding components of

\[
\begin{align*}
(0) + C_1 &= 0 \\
\sin(0) + C_2 &= 0 \\
-\cos(0) + C_3 &= 0
\end{align*}
\]

which implies \(C_1 = 0, C_2 = 0\) and \(C_3 = 1\). Thus the velocity function of the particle is \(\mathbf{v}(t) = \langle t, \sin t, 1 - \cos t \rangle\).

Then the position function of the particle is \(\mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int \langle t, \sin t, 1 - \cos t \rangle \, dt = \langle \frac{1}{2}t^2 + K_1, -\cos t + K_2, t - \sin t + K_3 \rangle\) for constants \(K_1, K_2, K_3\). Given the initial position of the particle, \(\mathbf{r}(0) = \langle 0, 0, 0 \rangle\), we find the values of the constants by solving the equations

\[
\begin{align*}
\frac{1}{2}(0)^2 + K_1 &= 0 \\
-\cos(0) + K_2 &= 0 \\
(0) - \sin(0) + K_3 &= 0
\end{align*}
\]

so that \(K_1 = 0, K_2 = 1, \) and \(K_3 = 0\). Thus

\[
\mathbf{r}(t) = \langle \frac{1}{2}t^2, 1 - \cos t, t - \sin t \rangle.
\]

Thus, the position of the particle at time \(t = \pi\) is \(\mathbf{r}(\pi) = \langle \frac{\pi^2}{2}, 2, \pi \rangle\).

(b) Write down an integral that will give the total distance traveled by the particle along this curve during the first second of travel.

**Solution:** The distance traveled by the particle along this curve during the first second is the arc length of the curve measured from time \(t = 0\) to time \(t = 1\), so it is \(L = \int_0^1 ||\mathbf{r}'(t)|| \, dt = \int_0^1 \sqrt{(\frac{1}{2}t^2 + 1 - \cos t)^2 + (\sin t)^2} \, dt = \int_0^1 \sqrt{\frac{1}{2}t^2 + 2 - 2\cos t} \, dt\).

(c) Bonus: Do not evaluate this integral, but show that it does not exceed \(\sqrt{3}\).

**Solution:** On the interval \(0 \leq t \leq 1, 0 \leq \cos t \leq 1\), so that \(t^2 \leq t^2 + 2 - 2\cos t \leq t^2 + 2 \leq 3\), since \(0 \leq t^2 \leq 1\) on \([0, 1]\). Thus \(L = \int_0^1 \sqrt{t^2 + 2 - 2\cos t} \, dt \leq \int_0^1 \sqrt{3} \, dt = \sqrt{3} [t]_0^1 = \sqrt{3}\) as claimed.